Scratch Work for Problem 0

Note that this scratch work **does not** constitute a valid proof since it is a form of backwards reasoning; we are starting with our desired result, and working backwards to obtain some suitable value of c and n_0.

The direct proofs are available on the section slides and the section handout solutions.

For each of the following, prove that $f(n) \in \mathcal{O}(g)$

(a) $f(n) = 7n$ $\quad g(n) = \frac{n}{10}$

Goal: $f(n) \leq c \cdot g(n) \Rightarrow 7n \leq c \cdot \frac{n}{10}$

Working backwards: $7n \leq c \cdot \dfrac{n}{10}$

$$7n \cdot 10 \leq c \cdot \frac{n}{10} \cdot 10$$

$$70n \leq c \cdot n$$

$$70 \leq c \quad \leftarrow \text{constraint for which our goal } 7n \leq c \cdot \frac{n}{10} \text{ holds}$$

Since n is not a variable in the above constraint, the constraint holds for all values of n. Therefore we can choose any value of n_0 as long as $c \geq 70$.

One such choice is $\boxed{n_0 = 1}$ and $\boxed{c = 70}$

(b) $f(n) = 1000$ $\quad g(n) = 3n^3$

Goal: $f(n) \leq c \cdot g(n) \Rightarrow 1000 \leq c \cdot 3n^3$

$f(n)$ is a constant function here, which makes things easy. We just need some choice of c and n_0 such that $3cn^3 \geq 1000$ for all $n \geq n_0$.

One such choice is $\boxed{n_0 = 1000}$ and $\boxed{c = 1}$

(c) $f(n) = 7n^2 + 3n$ $\quad g(n) = n^4$

Goal: $f(n) \leq c \cdot g(n) \Rightarrow 7n^2 + 3n \leq c \cdot n^4$

Property C.1: $n^4 \geq n^3 \geq n^2 \geq n$ when $n \geq 1$

If $n_0 = 1$, then for all $n \geq n_0 = 1$

$$7n^2 + 3n \leq c \cdot n^4$$

$$7n^2 + 3n \leq 7n^4 + 3n^4 \leq c \cdot n^4 \quad \leftarrow 7n^2 \leq 7n^4 \text{ and } 3n \leq 3n^4 \text{ by Property C.1}$$

$$10n^4 \leq c \cdot n^4$$

$$10 \leq c \quad \leftarrow \text{constraint for which our goal } 7n^2 + 3n \leq c \cdot n^4 \text{ holds}$$

The constraint holds when $n_0 = 1$. Therefore we can choose any value $c \geq 10$ as long as $n_0 = 1$.

One such choice is $\boxed{n_0 = 1}$ and $\boxed{c = 10}$

(d) $f(n) = n + 2n \log n$ $g(n) = n \log n$

Goal: $f(n) \leq c \cdot g(n) \Rightarrow n + 2n \log n \leq c \cdot n \log n$

Property D.1: $1 \leq \log n$ when $n \geq 2$ (recall that we always use log with base-2)

If $n_0 = 2$, then for all $n \geq n_0 = 2$

$$n + 2n \log n \leq c \cdot n \log n$$

$$n + 2n \log n \leq n \log n + 2n \log n \leq c \cdot n \log n \quad \leftarrow n \leq n \log n \text{ by Property D.1}$$

$$3n \log n \leq c \cdot n \log n$$

$$3 \leq c \quad \leftarrow \text{constraint for which our goal } n + 2n \log n \leq c \cdot n \log n \text{ holds}$$

The constraint holds when $n_0 = 2$. Therefore we can choose any value $c \geq 3$ as long as $n_0 = 2$.

One such choice is $\boxed{n_0 = 2}$ and $\boxed{c = 3}$